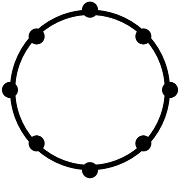
The Frog Problem

Huy Mai
Brandeis University
Gordon Rojas Kirby
Stanford University

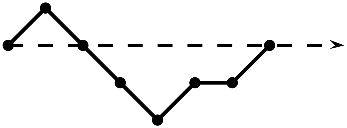
PSU Math REU
August 7, 2012

Preliminaries

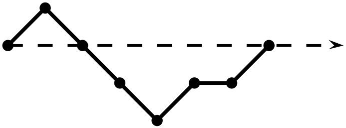


- Cycle graph, such that each frog has two neighbors
- Identical probability p for each frog at each iteration
- Restriction: A frog returns if it is more than one level above its neighbors

Preliminaries



Preliminaries



$\rightarrow (1, -1, -1, -1, 1, 0, 1, 0)$

- Represent Motzkin Path as a vector
- We call this a **state**
- The number of states are enumerated by the central trinomial coefficients
- **Our goal is to get a bound for the asymptotic speed S(K)**

Markov Chains

Definition

A **Markov Chain** is collection of random variables X_t, $t = 0, 1, ...,$ having the property that, given the present, the future is conditionally independent of the past.

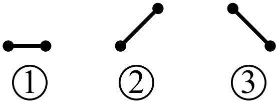
Markov Chains

Definition

A **Markov Chain** is collection of random variables X_t, $t = 0, 1, ...$, having the property that, given the present, the future is conditionally independent of the past.

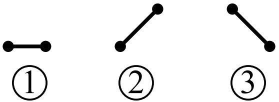
- For any finite K we have a Markov chain where the random variable represent transition probabilities between states

Example, $K = 2$



$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix} \end{array}$$

Example, $K = 2$



$$M = \begin{pmatrix} Q & \vdots & R \\ \cdots & & \cdots \\ 0 & \vdots & I_K \end{pmatrix}$$

For $K = 2$ we have

$$Q = \begin{pmatrix} q^2 & pq & pq \\ 0 & pq + q^2 & 0 \\ 0 & 0 & pq + q^2 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} p^2 & 0 & 0 \\ pq & p^2 & 0 \\ pq & 0 & p^2 \end{pmatrix}.$$

Example, $K = 2$

It can be easily checked that

$$(I - Q)^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2p-3}{p(p-2)} \\ \frac{1}{2p(1-p)} \\ \frac{1}{2p(1-p)} \end{pmatrix}$$

Example, $K = 2$

It can be easily checked that

$$(I - Q)^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2p-3}{p(p-2)} \\ \frac{1}{2p(1-p)} \\ \frac{1}{2p(1-p)} \end{pmatrix}$$

- We can do this for any fixed $K \in \mathbb{N}$, i.e. $S(K)$ exists
- In this example we see that the expected time spent in state 2 and state 3 is the same
- e.g. When $p = \frac{1}{2}$ we have $\begin{pmatrix} 8/3 \\ 2 \\ 2 \end{pmatrix}$

We need a new approach for when $K \to \infty$. Define

- $X_{n,k}$ are our random variables indexed by level and frog
- Indicator function $L_n(i,j)$
- $T(N,K) = \max_{k_1,\ldots,k_N}\{(\prod_{n=2}^{N} L_n(k_{n-1},k_n))\sum_{n=1}^{N} X_{n,k_n}\}$
- $r = E[\sum_{j=1}^{K} L_n(i,j)]$ for all i and n

Theorem (Chang and Nelson)

If the moment generating function of the time it takes for a frog to jump, $X_{n,k}$, exists for some finite $0 < \theta_0$

$$\phi(\theta) \stackrel{def}{=} E[e^{\theta X_{n,k}}] < \infty \quad \text{for } \theta \leq \theta_0$$

then the asymptotic speed $S(K)$ for all of the frogs to jump higher than some fixed level is bounded below $\frac{1}{t^*}$, where

$$t^* = \inf\{t \geq 1 | rm(t) < 1\}$$

and

$$m(t) = \inf_{0<\theta<\theta_0} \{e^{-\theta t} \phi(\theta)\}.$$

Sketch of Proof

Definition

A **Martingale** is a sequence of random variables $X_1, X_2, ...$ where the following is true

$$E[|X_n|] < \infty \quad \text{and} \quad E[X_{n+1}|X_1, ...X_n] = X_n$$

$$M_n(\theta) = \frac{1}{(r\phi(\theta))^n} \sum_{k_1=1}^{K} \cdots \sum_{k_N=1}^{K} \left(\prod_{m=2}^{n} L_{m-1}(k_{m-1}, k_m) \right) e^{\theta \sum_{m=1}^{n} X_{m,k_m}}$$

Sketch of Proof

Lemma

For the system as previously defined,

$$\frac{1}{S(K)} = \limsup_{N\to\infty} \frac{T(N,K)}{N} \le t^*, \quad a.s.$$

$$e^{\theta T(N,K)} = \max_{k_1,\ldots,k_N} \left\{ \prod_{n=2}^{N} L_n(k_{n-1}, k_n) e^{\theta \sum_{n=1}^{N} X_{n,k_n}} \right\} \le (r\phi(\theta))^N M_N(\theta).$$

Sketch of Proof

Markov's Inequality

$$P(X > t) \leq \frac{E[X]}{t}$$

$$\begin{aligned} P\left(\frac{T(N,K)}{N} > t\right) &= P\left(e^{\theta T(N,K)} > e^{\theta N t}\right) \\ &\leq e^{-\theta N t} E\left[e^{\theta T(N,K)}\right] \\ &\leq \frac{K}{r}\left(r e^{-\theta t} \phi(\theta)\right)^N \end{aligned}$$

Sketch of Proof

Thus, since $\sum_N P\left(\frac{T(N,K)}{N} > t\right) < \infty$ if $rm(t) < 1$. Hence we have

$$P\left(\limsup_{N\to\infty}\left(\frac{T(N,K)}{N} > t\right)\right) = 0$$

which implies

$$\limsup_{n\to\infty}\frac{T(N,K)}{N} \le t^*, \text{a.s}$$

Main Theorem

Theorem

Given K^2 iteations, the probability that all frogs have jumped above level N, where $N = aK^2$ and $a < \frac{-\ln q}{\ln r}$, is $1 - \alpha\beta^{K^2}$, where $\beta < 1$. That is,

$$P(T(N,K) \leq K^2) \stackrel{K\to\infty}{\longrightarrow} 1$$

Main Theorem

Theorem

Given K^2 iteations, the probability that all frogs have jumped above level N, where $N = aK^2$ and $a < \frac{-\ln q}{\ln r}$, is $1 - \alpha\beta^{K^2}$, where $\beta < 1$. That is,

$$P(T(N,K) \leq K^2) \xrightarrow{K \to \infty} 1$$

$$P(T(N,K) > K^2) = P(e^{\theta T(N,K)} > e^{\theta K^2})$$

$$\phi(\theta) = \frac{pe^{\theta}}{1 - qe^{\theta}}$$

Acknowledgements

We would like to thank Misha Guysinski and the support and funding we received from the PSU Math REU

Huy Mai
Brandeis University
huymai@brandeis.edu

Gordon Rojas Kirby
Stanford University
girkirby@stanford.edu